Introduction to Orthogonal Functions and Eigenfunction Expansions

Goal of these notes

Function sets can form vector spaces and the notions of vectors and matrix operations – orthogonality, basis sets, eigenvalues, can be carried over into analysis of functions that are important in engineering applications. (In dealing with functions we have eigenfunctions in place of eigenvectors.) These notes link the previous notes on vector spaces to the application to functions.

The eigenfunctions with which we will be dealing are solutions to differential equations. Differential equations, both ordinary and partial differential equations, are an important part of engineering analysis and play a major role in engineering analysis courses. We will begin with a brief review of ordinary differential equations. We will then discuss power series solutions to differential equations and apply this technique to Bessel's differential equation. The series solutions to this equation, known as Bessel functions, usually occur in cylindrical geometries in the solution to the same problems that produce sines and cosines in rectangular geometries.

We will see that Bessel functions, like sines and cosines, form a complete set so that any function can be represented as an infinite series of these functions. We will discuss the Sturm-Loiuville equation, which is a general approach to eigenfunction expansions, and show that sines, cosines, and Bessel functions are special examples of functions that satisfy the Sturm-Liouville equation.

The Bessel functions are just one example of special functions that arise as solutions to ordinary differential equations. Although these special functions are less well known than sines and cosines, the idea that these special functions behave in a similar general manner to sines and cosines in the solution of engineering analysis problems, is a useful concept in applying these functions when the problem you are solving requires their use.

These notes begin by reviewing some concepts of differential equations before discussing power series solutions and Frobenius method for power series solutions of differential equations. We will then discuss the solution of Bessel's equation as an example of Frobenius method. Finally, we will discuss the Sturm-Liouville problem and a general approach to special functions that form complete sets.

What is a differential equation?

A differential equation is an equation, which contains a derivative. The simplest kind of a differential equation is shown below:

$$\frac{dy}{dx} = f(x) \qquad with \qquad y = y_0 \ at \ x = x_0 \tag{1}$$

In general, differential equations have an infinite number of solutions. In order to obtain a unique solution one or more initial conditions (or boundary conditions) must be specified. In the above example, the statement that $y = y_0$ at $x = x_0$ is an initial condition. (The difference between initial and boundary conditions, which is really one of naming, is discussed below.) The differential equation in [1] can be “solved” as a definite integral.

$$y - y_0 = \int_{x_0}^{x} f(x)dx \qquad [2]$$

The definite integral can be either found from a table of integrals or solved numerically, depending on f(x). The initial (or boundary) condition ($y = y_0$ at $x = x_0$) enters the solution directly. Changes in the values of y_0 or x_0 affect the ultimate solution for y.

A simple change – making the right hand side a function of x and y, f(x,y), instead of a function of x alone – gives a much more complicated problem.

$$\frac{dy}{dx} = f(x,y) \qquad with \qquad y = y_0 \ at \ x = x_0 \qquad [3]$$

We can formally write the solution to this equation just as we wrote equation [2] for the solution to equation [1].

$$y - y_0 = \int_{x_0}^{x} f(x,y)dx \qquad [4]$$

Here the definite integral can no longer be evaluated simply. Thus, alternative approaches are needed. Equation [4] is used in the derivation of some numerical algorithms. The (unknown) exact value of f(x,y) is replaced by an interpolation polynomial which is a function of x only.

In the theory of differential equations, several approaches are used to provide analytical solutions to the differential equations. Regardless of the approach used, one can always check to see a proposed solution is correct by substituting a proposed solution into the original differential equation and determining if the solution satisfies the initial or boundary conditions.

Ordinary differential equations involve functions, which have only one independent variable. Thus, they contain only ordinary derivatives. **Partial differential equations** involve functions with more than one independent variable. Thus, they contain partial derivatives. The abbreviations ODE and PDE are used for ordinary and partial differential equations, respectively.

In an ordinary differential equation, we are usually trying to solve for a function, y(x), where the equation involves derivatives of y with respect to x. We call y the **dependent variable** and x the **independent variable**.

The **order of the differential equation** is the order of the highest derivative in the equation. Equations [1] and [3] are first-order differential equations. A differential equation with first, second and third order derivatives only would be a third order differential equation.

In a **linear differential equation**, the terms involving the dependent variable and its derivatives are all linear terms. The independent variable may have nonlinear terms. Thus $x^3d^2y/dx^2 + y = 0$ is a linear, second-order differential equation. ydy/dx + sin(y) = 0 is a nonlinear first-order differential equation. (Either term in this equation – ydy/dx or sin(y) would make the differential equation nonlinear.)

Differential equations need to be accompanied by **initial or boundary conditions**. An n^{th} order differential equation must have n initial (or boundary) conditions in order to have a unique solution. Although initial and boundary conditions both mean the same thing, the term "initial conditions" is usually used when all the conditions are specified at one initial point. The term "boundary conditions" is used when the conditions are specified at two different values of the independent variable. For example, in a second order differential equation for y(x), the specification that y(0) = a and y'(0) = b, would be called two initial conditions. The specification that y(0) = c and y(L) = d, would be called boundary conditions. The initial or boundary conditions can involve a value of the variable itself, lower-order derivatives of the variable, or equations containing both values of the dependent variable and its lower-order derivatives.

Some simple ordinary differential equations

From previous courses, you should be familiar with the following differential equations and their solutions. If you are not sure about the solutions, just substitute them into the original differential equation.

$$\frac{dy}{dt} = -ky \qquad with \qquad y = y_0 \; at \; t = t_0 \qquad \Rightarrow \qquad y = y_0 e^{-k(t-t_0)} \qquad [5]$$

$$\frac{d^2y}{dx^2} = -k^2 y \qquad \Rightarrow \qquad y = A\sin(kx) + B\cos(kx) \qquad [6]$$

$$\frac{d^2y}{dx^2} = k^2 y \qquad \Rightarrow \qquad y = A\sinh(kx) + B\cosh(kx) = A'e^{kx} + B'e^{-kx} \qquad [7]$$

In equations [6] and [7] the constants A and B (or A' and B') are determined by the initial or boundary conditions. Note that we have used t as the independent variable in equation [5] and x as the independent variable in equations [6] and [7].

There are four possible functions that can be a solution to equation [6]: sin(kx), cos(kx), e^{ikx}, and e^{-ikx}, where $i^2 = -1$. Similarly, there are four possible functions that can be a solution to equation [7]: sinh(kx), cosh(kx), e^{kx}, and e^{-kx}. In each of these cases the four possible solutions are not linearly independent.[1] The minimum number of functions with which all solutions to the differential equation can be expressed is called a basis set for the solutions. The solutions shown above for equations [6] and [7] are basis sets for the solutions to those equations.

One final solution that is useful is the solution to general linear first-order differential equation. This equation can be written as follows.

$$\frac{dy}{dx} + f(x)y = g(x) \qquad [8]$$

This equation has the following solution, where the constant, C, is determined from the initial condition.

[1] We have the following equations among these various functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2} \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$y = e^{-\int f(x)dx}\left[C + \int\left(g(x)e^{\int f(x)dx}\right)dx\right] \qquad [9]$$

Power series solutions of ordinary differential equations

The solution to equation [6] is composed of a sine and a cosine term. If we consider the power series for each of these, we see that the solution is equivalent to the following power series.

$$y = A\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) + B\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = A\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} + B\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$
[10]

We are interested in seeing if we can obtain such a solution directly from the differential equation. A proof beyond the level of these notes can be used to show that the following differential equation $\frac{d^2y(x)}{dx^2} + p(x)\frac{dy(x)}{dx} + q(x)y = r(x)$ has power series solutions, y(x) in a region, R, around x = a, provided that p(x), q(x) and r(x) can be expressed in a power series in some region about x = a. Functions that can be represented as a power series are called analytic functions.[2]

The power series solution of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$ requires that the three functions p(x), q(x) and r(x) can be represented as power series. Then we assume a solution of the following form.

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad [11]$$

Here the a_n are unknown coefficients. We can differentiate this series twice to obtain.

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \qquad and \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \qquad [12]$$

Substituting equations [11] and [12] into our original differential equation gives the following result.

$$\sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2} + p(x)\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} + q(x)\sum_{n=0}^{\infty} a_n (x - x_0)^n = r(x) \qquad [13]$$

We then set the coefficients of each power of x on both sides of the equation to be equal to each other. This gives an equation that we can used to solve for the unknown a_n coefficients in terms of one or more coefficients like a_0 and a_1, which are used to determine the initial conditions. This

[2] See the brief discussion of power series in Appendix A for more basic information on power series.

is best illustrated by using equation [6], $\frac{d^2y}{dx^2}+k^2y=0$, as an example. Here we have p(x) = 0, q(x) = k^2, and r(x) = 0, so for this example, equation [13] becomes.

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2}+k^2\sum_{n=0}^{\infty} a_n(x-x_0)^n=0 \qquad [14]$$

The only way to assure that equation [14] is satisfied is to have the coefficient of each power of (x – x_0) vanish. We get the power series solution by setting the coefficients of each power of (x – x_0) equal to zero. This task is simplified if we collect all the terms in equation [14] into a single sum. To do this, we note that the first two terms (n = 0 and n = 1) in the first sum of equation [14] are zero. We can thus start the sum at n = 2. Next, we can change the index on this sum from n to a new index, m = n – 2. Finally, we can combine the two sums, even though they have different summation indices, because these indices are dummy indices and the both limits on each summation are the same.

These steps give the following result.

$$0=\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2}+k^2\sum_{n=0}^{\infty} a_n(x-x_0)^n=\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$
$$+k^2\sum_{n=0}^{\infty} a_n(x-x_0)^n=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-x_0)^m+k^2\sum_{n=0}^{\infty} a_n(x-x_0)^n$$
$$=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n+k^2\sum_{n=0}^{\infty} a_n(x-x_0)^n=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+k^2a_n\right](x-x_0)^n$$
[15]

The last sum in equation [15] equals zero only if the coefficient of (x – x_0)^n vanishes for each n. This gives the following relationship among the unknown coefficients.

$$(n+2)(n+1)a_{n+2}+k^2a_n=0 \qquad or \qquad a_{n+2}=-\frac{k^2a_n}{(n+2)(n+1)} \qquad [16]$$

This gives us an equation for a_{n+2} in terms of the coefficient previously found for a_n. We cannot use this equation to find a_0 or a_1, so we assume that these coefficients will be determined by the initial conditions. However, once we know a_0, we see that we can find all the even numbered coefficients as follows

$$a_2=-\frac{k^2a_0}{(0+2)(0+1)}=-\frac{k^2a_0}{2} \qquad a_4=-\frac{k^2a_2}{(2+2)(2+1)}=-\frac{k^2\left(-\frac{k^2a_0}{2}\right)}{(2+2)(2+1)}=\frac{k^4a_0}{4\cdot 3\cdot 2}$$
[17]

Continuing in this fashion we see that the general pattern for a subscript whose coefficient is an even number is the following.

$$a_n = \frac{(-1)^{n/2} k^n a_0}{(n)!} \qquad n \text{ even} \qquad [18]$$

We can verify this general result by obtaining an equation for a_{n+2}. This is done by replacing n in equation by n+2 to give $a_{n+2} = \frac{(-1)^{(n+2)/2} k^{n+2} a_0}{(n+2)!}$. Next we substitute this equation and equation [18] into equation [16] to see if we get the correct result for the ratio a_{n+2}/a_n.[3]

$$\frac{a_{n+2}}{a_n} = -\frac{k^2}{(n+2)(n+1)} = \frac{\dfrac{(-1)^{(n+2)/2} k^{n+2} a_0}{(n+2)!}}{\dfrac{(-1)^{n/2} k^n a_0}{n!}} = \frac{-k^2 n!}{(n+2)!} = -\frac{k^2}{(n+2)(n+1)} \qquad [19]$$

We see that the ratio a_{n+2}/a_n that we computed using our general equation for a_n from equation [18] is the same as the value for this ratio that we started with in equation [16]. We thus conclude that equation [18] gives us a correct solution for a_n when n is even. We can handle the recurrence for a_n, when n is an odd number in the same way that we just did for even n. We start by finding a_3 and a_5 in terms of a_1.

$$a_3 = -\frac{k^2 a_1}{(1+2)(1+1)} = -\frac{k^2 a_1}{3 \cdot 2} \qquad a_5 = -\frac{k^2 a_3}{(3+2)(3+1)} = -\frac{k^2\left(-\dfrac{k^2 a_1}{3\cdot 2}\right)}{5\cdot 4} = \frac{k^4 a_1}{5\cdot 4\cdot 3\cdot 2}$$

[20]

We see that this recurrence will lead to a general equation of the following form.

$$a_n = \frac{(-1)^{(n-1)/2} k^{n-1} a_1}{(n)!} \qquad n \text{ odd} \qquad [21]$$

As before, we can check this general relationship by obtaining an expression for a_{n+2} and showing that the ratio a_{n+2}/a_n as computed from equation [21] satisfies equation [16]. This check is left as an exercise for the reader.

Now that we have expressions for a_n in terms of the initial values a_0 and a_1 we can substitute these expressions (in equations [18] and [21]) into our proposed general power series solution for our differential equation from equation [11].

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[3] If you are not familiar with the cancellation of factorials, see the discussion in Appendix B.

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{\substack{n=0 \\ even\ n}}^{\infty} a_n (x - x_0)^n + \sum_{\substack{n=1 \\ odd\ n}}^{\infty} a_n (x - x_0)^n =$$

$$\sum_{\substack{n=0 \\ even\ n}}^{\infty} \frac{(-1)^{n/2} k^n a_0}{(n)!} (x - x_0)^n + \sum_{\substack{n=1 \\ odd\ n}}^{\infty} \frac{(-1)^{(n-1)/2} k^{n-1} a_1}{(n)!} (x - x_0)^n =$$

$$a_0\left[1 - \frac{[k(x - x_0)]^2}{2!} + \frac{[k(x - x_0)]^4}{4!} - \cdots\right] + a_1\left[k(x - x_0) - \frac{[k(x - x_0)]^3}{3!} + \frac{[k(x - x_0)]^5}{5!} - \cdots\right] \quad [22]$$

Thus the series multiplied by a_0 and a_1 are seen to be the series for cos[k(x – x_0)] and sin[k(x – x_0)], respectively. This is the expected solution of the differential equation $\frac{d^2y}{dx^2} + k^2 y = 0$. Although this differential equation has a solution in terms of sines and cosines, the basic power series methods can be used for equations that do not have a conventional solution.

Summary of power series solutions of ordinary differential equations

We can solve a differential equation like $\frac{d^2y(x)}{dx^2} + p(x)\frac{dy(x)}{dx} + q(x)y = r(x)$, using the power series method, provided that p(x), q(x) and r(x) are analytic at a point x_0 where we want the solution. Such a solution is obtained in the following steps.

- Write the solution for y(x) as a power series in unknown coefficients a_n as shown in equation [11].
- Differentiate the power series two times to get the derivatives required in the differential equation; see equation [12] for the results of this differentiation.
- Obtain power series expansions for p(x), q(x) and r(x), if these are not constants or simply polynomials.
- Substitute the power series for y(x), y'(x), y"(x), p(x), q(x) and r(x) into the differential equation for the problem.
- Rewrite the resulting equation to group terms with common powers of x – x_0.
- Set the coefficients of each power of x – x_0 equal to zero. This should produce an equation that relates neighboring values of the unknown coefficients a_n.
- Use the equation found in the previous step to relate coefficients with higher subscripts to those with lower subscripts. The first few coefficients, e.g., a_0, a_1, etc., will not be known. (These will be determined by the initial conditions on the differential equation.)
- Examine the equation relating the coefficients and try to obtain a general equation for each a_n in terms of the unknown coefficients a_0, a_1, etc.
- Substitute the general expression for a_n into the original power series for y(x). This is the final power series solution.

Frobenius method for solution of ordinary differential equations

The Frobenius method is used to solve the following differential equation.

$$\frac{d^2 y(x)}{dx^2} + \frac{b(x)}{x}\frac{dy(x)}{dx} + \frac{c(x)}{x^2} y = 0 \qquad [23]$$

In this equation b(x) and c(x) are analytic at x = 0. Note that the conventional power series method cannot be used for this equation because the coefficients of dy/dx and y are not analytic at x = 0.

In the Frobenius method the solution is written as follows.

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad [24]$$

Here the a_n are unknown coefficients and the value of r is also unknown. The value of r is determined during the solution procedure so that $a_0 \neq 0$. We can differentiate the series for y(x) in equation [24] two times to obtain.

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \qquad and \qquad \frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \qquad [25]$$

We assume that b(x) and c(x) are constants, simple polynomials or series expressions that have the following general forms.

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \qquad c(x) = \sum_{n=0}^{\infty} c_n x^n \qquad [26]$$

We can substitute equations [24], [25], and [26], into equation [23] and multiply the result by x^2 to obtain the following equation.

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x\left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

[27]

We can combine the x^2 and x terms outside the sums with the x terms inside the sums as follows.

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \qquad [28]$$

We can then write out the first few terms in each series or product of series to give

$$\begin{aligned} & r(r-1)a_0 x^r + (1+r) r a_1 x^{r+1} + (2+r)(1+r) a_2 x^{r+2} + \cdots \\ & + b_0 r a_0 x^r + \left[b_1 r a_0 + b_0 (1+r) a_1\right] x^{r+1} + \left[b_2 r a_0 + b_1 (1+r) a_1 + b_0 (2+r) a_2\right] x^{r+2} + \cdots \\ & + c_0 a_0 x^r + \left[c_1 a_0 + c_0 a_1\right] x^{r+1} + \left[c_2 a_0 + c_1 a_1 + c_0 a_2\right] x^{r+2} + \cdots = 0 \end{aligned} \qquad [29]$$

As in the conventional power series solution, we require the coefficient of each term in the power series to vanish to satisfy equation [28] or [29]. Starting with the lowest power of x, x^r, we require that the following coefficient be zero.

$$\left[r(r-1)+b_0 r+c_0\right]a_0=0 \qquad [30]$$

In this case we want to keep $a_0 \neq 0$, thus we require that

$$r(r-1)+b_0 r+c_0=0 \qquad [31]$$

This yields a quadratic equation in r which is called the **indicial equation**. We find two possible values of r from the conventional solution of the quadratic equation.

$$r=\frac{1-b_0 \pm \sqrt{(1-b_0)^2-4c_0}}{2} \qquad [32]$$

The original solution in equation [24] is a basis for all solutions if the two values of r found from equation [23] are different and their difference is not an integer. In that case we have two solutions which can form a basis for all solutions to equation [23]. These are

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n \qquad \textit{and} \qquad y_2(x)=x^{r_2}\sum_{n=0}^{\infty}A_n x^n \qquad [33]$$

The coefficients for the two solutions, a_n for $y_1(x)$ and A_n for $y_2(x)$, are different. These coefficients are found in the same way that the coefficients were found in the usual power series equation, once the values of r are determined. If there is a double root for r or if the two values of r differ by an integer, it is necessary to have a second solution that has a different form. For a double root, we have the two following solutions.

$$y_1(x)=x^{r}\sum_{n=0}^{\infty}a_n x^n \qquad \textit{and} \qquad y_2(x)=y_1(x)\ln(x)+\sum_{n=1}^{\infty}A_n x^n \qquad [34]$$

If the two roots, r_1 and r_2 differ by an integer, the two possible solutions are written as follows.

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n \qquad \textit{and} \qquad y_2(x)=ky_1(x)\ln(x)+x^{r_2}\sum_{n=0}^{\infty}A_n x^n \qquad [35]$$

In the last expression, the roots are defined such that $r_1 > r_2$; the value of k may be zero in this case. Note that the a_n coefficients and the A_n coefficients are different; also for the double root, the summation for $y_2(x)$ starts at n = 1 instead of n = 0.

In the next section, we examine the solution of Bessel's equation using the Frobenius method. This serves both as the derivation of the series expansions for the various Bessel functions and as an illustration of the Frobenius method.

Application of Frobenius Method to Bessel's equation

Bessel's equation arises when various engineering phenomena are modeled in cylindrical coordinates. This equation for y(x) with ν as a known parameter in the equation has the following form.

$$\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + \frac{x^2 - \nu^2}{x^2}y = 0 \quad [36]$$

We see that this is the general form given in equation [23] with b(x) = 1 and c(x) = $x^2 - \nu^2$. Recall the basic solution format for Frobenius method from equation [24], copied below.

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad [24]$$

The derivatives of this solution were given in equation [25], also copied below.

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad and \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \quad [25]$$

If we multiply equation [36] by x^2 and substitute equations [24] and [25] we obtain the following result.

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \left(x^2 - \nu^2\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0 \quad [37]$$

Except for the final sum that is multiplied by x^2, we can combine all three summation operators into a single sum.

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (n+r) - \nu^2\right]a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0 \quad [38]$$

We can simplify the coefficient in the first summation.

$$(n+r)(n+r-1) + (n+r) - \nu^2 = (n+r)^2 - (n+r) + (n+r) - \nu^2 = (n+r)^2 - \nu^2 \quad [39]$$

We can define a new index for the second sum, m = n+2. This allows us to write the second sum as follows.

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{m=2}^{\infty} a_{m-2} x^{m+r} \quad [40]$$

We can use equations [39] and [40] to rewrite equation [38] as follows.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (n+r) - \nu^2\right]a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = \\
&\sum_{n=0}^{\infty}\left[(n+r)^2 - \nu^2\right]a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = \\
&\left(r^2 - \nu^2\right)a_0 x^r + \left[(1+r)^2 - \nu^2\right]a_1 x^{1+r} + \sum_{n=2}^{\infty}\left\{\left[(n+r)^2 - \nu^2\right]a_n + a_{n-2}\right\}x^{n+r} = 0
\end{aligned} \quad [41]$$

The indicial equation is the equation that sets the coefficient of a_0 equal to zero. For Bessel's equation, the indicial equation is $r^2 - \nu^2 = 0$, whose solution is $r = \pm\nu$. The first Frobenius method solution will be the same regardless of the value of ν. However, if $\nu = 0$, this is a double root and if ν is an integer, the two roots of the indicial equation will differ by an integer. The first solution will be the same in all three cases, so we will obtain this solution and defer the consideration of the second solution. We will take the root $r = \nu$, that gives a positive value for r. We will assume that ν is positive. Substituting $r = \nu$, into the power series solution in equation [41] gives

$$\left[(1+\nu)^2 - \nu^2\right]a_1 x^{1+\nu} + \sum_{n=2}^{\infty}\left\{\left[(n+\nu)^2 - \nu^2\right]a_n + a_{n-2}\right\}x^{n+\nu} = 0 \qquad [42]$$

Each coefficient in this equation (for each power of x) must equal zero for series sum on the left-hand side of equation [42] to be zero. The coefficient of the $x^{1+\nu}$ term will not vanish unless $a_1 = 0$. Thus, we conclude that a_1 is zero. The remaining powers of x in equation [42] have a common equation for the coefficient. When we set this equation to zero, we obtain the following result.

$$\left[(n+\nu)^2 - \nu^2\right]a_n + a_{n-2} = \left[n^2 + 2n\nu + \nu^2 - \nu^2\right]a_n + a_{n-2} = n(n+2\nu)a_n + a_{n-2} = 0 \qquad [43]$$

We can solve this equation to obtain a recurrence relationship that gives a_n in terms of a_{n-2}.

$$a_n = \frac{-a_{n-2}}{n(n+2\nu)} \qquad [44]$$

According to equation [44], if $a_n = 0$, then $a_{n+2} = 0$. Since we have shown that a_1 must be zero, we conclude that all values of a_n with an odd subscript must be zero. Since the further application of equation [44] will be for even subscripts only, we change the index from n to m where n = 2m. This allows us to rewrite equation [44] as follows.

$$a_{2m} = \frac{-a_{2m-2}}{2m(2m+2\nu)} = \frac{-a_{2m-2}}{2^2 m(m+\nu)} \qquad [45]$$

We will take a_0 as an unknown coefficient that is determined by the initial conditions on the differential equation. We can then use equation [45] to write the first few (even-numbered) coefficients in terms of a_0. Setting m = 1 in equation [45] gives us a_2.

$$a_2 = \frac{-a_{2(1)-2}}{2^2(1)(1+\nu)} = \frac{-a_0}{2^2(1+\nu)} \qquad [46]$$

Next we set m = 2 in equation [45] to obtain a_4 in terms of a_2, and then use equation [46] to get a_4 in terms of a_0.

$$a_4 = \frac{-a_{2(2)-2}}{2^2(2)(2+\nu)} = \frac{-a_2}{2^2(2)(2+\nu)} = \frac{-\left(\dfrac{-a_0}{2^2(1+\nu)}\right)}{2^2(2)(2+\nu)} = \frac{a_0}{2^4(2)(2+\nu)(1+\nu)} \qquad [47]$$

Next, we use m = 3 in equation [45] to compute a_6 by which point we should be able to see the pattern in the a_{2m} coefficients.

$$a_6 = \frac{-a_{2(3)-2}}{2^2(3)(3+\nu)} = \frac{-a_4}{2^2(3)(3+\nu)} = \frac{-\left(\frac{a_0}{2^4(2)(2+\nu)(1+\nu)}\right)}{2^2(3)(3+\nu)} = \frac{-a_0}{2^6(3)(2)(3+\nu)(2+\nu)(1+\nu)} \quad [48]$$

We see a pattern emerging that we can summarize in the following general equation for the coefficients in the power series solution.

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(m+\nu)(m-1+\nu)\cdots(2+\nu)(1+\nu)} \quad [49]$$

This is the general relationship for the coefficients in the first solution. We now have to look at individual cases of noninteger ν, integer ν, and ν = 0.

First Frobenius method solution for integer ν – We first consider the case of integer ν, for which the parameter, ν, is conventionally represented as n. (The symbol ν is reserved for cases where this parameter is not an integer.) Setting n = ν for cases where n is an integer gives the following version of equation [49].

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(m+n)(m-1+n)\cdots(2+n)(1+n)} \quad [50]$$

Since a_0 is an unknown constant, which has different values for different problems as determined by the boundary conditions for the problem, we can redefine a_0 as follows. (This is a convention used to obtain an equation that is used for computation and tabulation of Bessel functions.)

$$a_0 = A\frac{1}{2^n n!} \quad [49]$$

Now, A is the constant that is selected to fit the boundary conditions. With this substitution, we can write equation [48] as follows.

$$a_{2m} = \frac{(-1)^m A}{2^{2m+n} m!\,(m+n)(m-1+n)\cdots(2+n)(1+n)n!} = \frac{(-1)^m A}{2^{2m+n} m!\,(m+n)!} \quad [50]$$

We can substitute this expression for a_{2m} into the power series equation [24] proposed for the solution with the value of r in that equation set to the solution of the indicial equation, r = ν, using n in place of ν for integer values of ν.

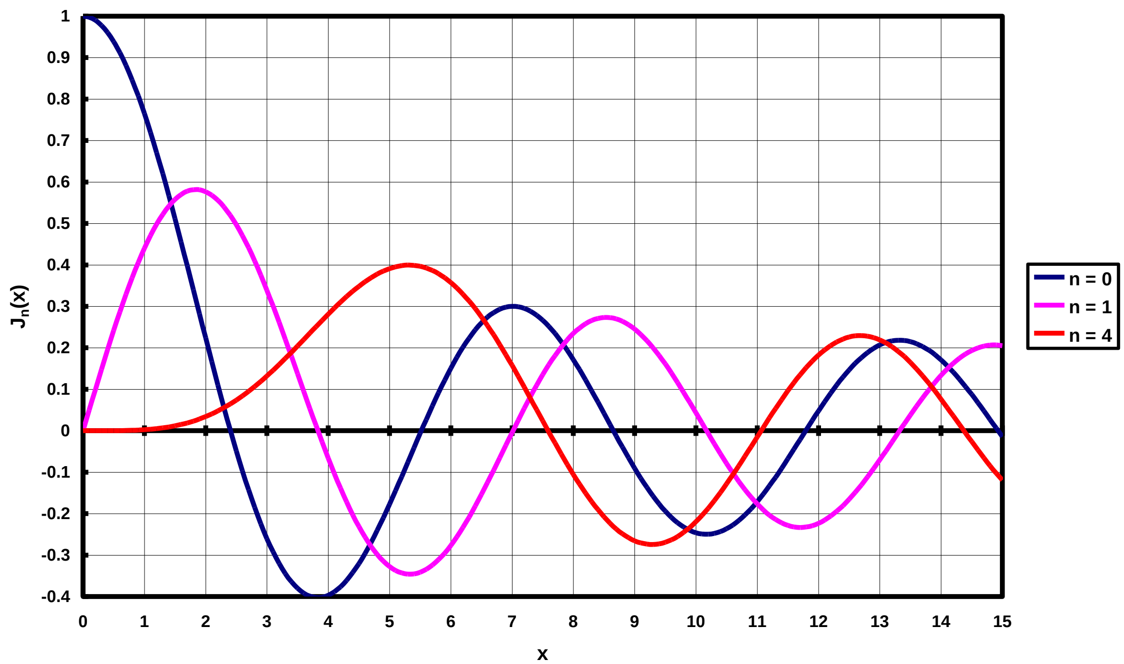
$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = x^n \sum_{m=0}^{\infty} a_{2m} x^{2m} = Ax^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(m+n)!} \quad [51]$$

The **Bessel function of the first kind of integer order n, $J_n(x)$** is defined by this equation, with the arbitrary constant omitted. (This is the same practice as ignoring the multiplicative constant in the sine and cosine solutions to differential equations and simply tabulating the sine and cosine.) Thus, we write

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(m+n)!} \qquad [52]$$

Plots of these Bessel functions for some low values of n are shown below. Note that $J_0(0) = 1$ while $J_n(0) = 0$ for all n > 0.

Bessel Functions of the First Kind for Integer Orders



First Frobenius method solution for noninteger n – The only change necessary when we consider noninteger ν, is that we do not have the factorials used above in the case of integer ν = n. Instead, we use the gamma function in the definition of a_0. (See appendix C for background on the gamma function and its ability to generalize factorial relationships to noninteger values.) That is we replace equation [49] by the following equation for noninteger ν.

$$a_0 = A\frac{1}{2^\nu \Gamma(\nu+1)} \qquad [53]$$

Substituting this expression for a_0 into equation [49], and using equation [C-3] from Appendix C on gamma functions gives the following result.

$$a_{2m} = \frac{(-1)^m A}{2^{2m+\nu} m!\,(m+\nu)(m-1+\nu)\cdots(2+\nu)(1+\nu)\Gamma(\nu+1)} = \frac{(-1)^m A}{2^{2m+\nu} m!\,\Gamma(\nu+m+1)} \qquad [54]$$

We substitute this equation for a_{2m} (without the arbitrary constant, A) into the general power series solution from equation [24] to define the **Bessel function of the first kind of (noninteger) order ν, $J_\nu(x)$**.

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\,\Gamma(m+\nu+1)} \quad [55]$$

Note that this definition for noninteger ν is the same as equation [52] for integer ν = n, since equation [C-6] shows that (m + n)! = Γ(m + n + 1).

Second Frobenius method solution – Now that we have the first solution for integer and noninteger ν, including ν = 0, we have to find another linearly independent solution to Bessel's equation to form a basis for all possible solutions to this equation. This requires us to consider each case separately. The simplest case is when ν is not an integer. Here the values of $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent and we can write a solution to Bessel's equation as follows.

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \quad [56]$$

Here A and B are arbitrary constants determined by the initial conditions on the original differential equation. The values of $J_{-\nu}(x)$ are found from equation [55].

When ν is an integer (with the usual notation that integer ν = n), we can use equation [52] to show that the Bessel functions for order n and order –n are linearly dependent. These two orders of Bessel functions are simply related as follows.

$$J_{-n}(x) = (-1)^n J_n(x) \quad [57]$$

Second Frobenius method solution for integer ν = n – To find a second solution for integer ν, we have to separately consider the special case where ν = n = 0. In this case, where the indicial equation had a double root, the second solution is shown in equation [34]. To get the second solution for Bessel's equation with n = 0, we modify equation [34] by setting the first solution to $J_0(x)$ and the value of r in equation [34] to the value of the double root, r = 0. This gives the equation shown below for the second solution

$$y_2(x) = J_0(x)\ln(x) + \sum_{m=1}^{\infty} A_m x^m \quad [58]$$

For the more general case of integer ν = n ≠ 0, the second solution is given by equation [35]. Since the two roots of the indicial equation are r = ±ν = ±n for integer ν, and we used r = +n in the original solution for $J_n(x)$, we have to use the second root, r = -n here. Using this value for r and $J_n(x)$ for the original solution, $y_1(x)$, equation [35] becomes.

$$y_2(x) = kJ_n(x)\ln(x) + x^{-n}\sum_{m=0}^{\infty} A_m x^m = kJ_n(x)\ln(x) + \sum_{m=0}^{\infty} A_m x^{m-n} \quad [59]$$

Equations [58] and [59] use m as the summation index in place of n, which was used in equations [34] and [35]. This was done to avoid confusion with the use of n to represent the order of the Bessel function. Since the summation index is a dummy index, we can use any symbol we want for this index.

We can do the initial part of the analysis for general integer ν = n, where n may or may not be zero. To do this we will write Bessel's equation [36] as follows, substituting n for ν.

$$x^2 \frac{d^2 y(x)}{dx^2} + x\frac{dy(x)}{dx} + \left(x^2 - n^2\right)y = 0 \qquad [60]$$

We will also use equation [59] as the second solution. When we do this, we must remember to set k = 1 and to set the lower limit of the summation equal to 1 to make the equation for general n in [59] correspond to equation [58] for n = 0. (These two changes, plus setting n = 0, will convert the solution in [59] to the special case of n = 0 in equation [58].)

Taking the first derivative of equation [59] gives

$$\frac{dy_2(x)}{dx} = k\frac{dJ_n(x)}{dx}\ln(x) + kJ_n(x)\frac{d\ln(x)}{dx} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n-1} \qquad [61]$$

Substituting 1/x for dln(x)/dx and taking the second derivative gives.

$$\frac{dy_2^2(x)}{dx^2} = k\frac{dJ_n^2(x)}{dx^2}\ln(x) - \frac{kJ_n(x)}{x^2} + \frac{2k}{x}\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n-2} \qquad [62]$$

Substituting equations [59], [61], and [62] into equation [60] gives the following result.

$$\begin{gathered} x^2 \frac{d^2 y_2(x)}{dx^2} + x\frac{dy_2(x)}{dx} + \left(x^2 - n^2\right)y_2 = \\ x^2\left[k\frac{dJ_n^2(x)}{dx^2}\ln(x) - \frac{kJ_n(x)}{x^2} + \frac{2k}{x}\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n-2}\right] \\ + x\left[k\frac{dJ_n(x)}{dx}\ln(x) + kJ_n(x)\frac{d\ln(x)}{dx} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n-1}\right] \\ + \left(x^2 - n^2\right)\left[kJ_n(x)\ln(x) + \sum_{m=0}^{\infty}A_m x^{m-n}\right] = 0 \end{gathered} \qquad [63]$$

We can rearrange this equation by multiplying each term in braces by the factors outside the braces, substituting 1/x for dln(x)/dx, and collecting all the terms multiplied by k ln(x).

$$\begin{gathered} k\ln(x)\left[x^2 \frac{dJ_n^2(x)}{dx^2} + x\frac{dJ_n(x)}{dx} + \left(x^2 - n^2\right)J_n(x)\right] - kJ_n(x) + kJ_n(x) \\ + 2kx\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n} \\ + \left(x^2 - n^2\right)\sum_{m=0}^{\infty}A_m x^{m-n} = 0 \end{gathered} \qquad [64]$$

The terms in the top row all vanish. The last two terms in this row obviously cancel. The terms multiplied by k ln(x) are just Bessel's equation (see equation [60]) with $J_n(x)$ as the dependent variable. However, $J_n(x)$ is a solution to Bessel's equation. Thus equation [60] tells us that the terms multiplying k ln(x) sum to zero.

We can consider the remaining terms in equation [64] as follows. First, we use equation [52] for J_n to evaluate the derivative term.

$$2kx\frac{dJ_n(x)}{dx} = 2kx\frac{d}{dx}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+n}}{2^{2m+n} m!\,(m+n)!}\right\} =$$
$$2kx\sum_{m=0}^{\infty}\frac{(-1)^m (2m+n)x^{2m+n-1}}{2^{2m+n} m!\,(m+n)!} = 2k\sum_{m=0}^{\infty}\frac{(-1)^m (2m+n)x^{2m+n}}{2^{2m+n} m!\,(m+n)!} \quad [65]$$

We can rewrite the summation terms in equation [64] as follows.

$$\sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n} + \left(x^2 - n^2\right)\sum_{m=0}^{\infty}A_m x^{m-n} =$$
$$\sum_{m=0}^{\infty}\left[(m-n)(m-n-1) + (m-n) - n^2\right]A_m x^{m-n} + \sum_{m=0}^{\infty}A_m x^{m-n+2} = \quad [66]$$
$$\sum_{m=0}^{\infty}m(m-2n)A_m x^{m-n} + \sum_{m=0}^{\infty}A_m x^{m-n+2}$$

We can now substitute equations [65] and [66] into equation [64], after setting the first row of [64] equal to zero.

$$2k\sum_{m=0}^{\infty}\frac{(-1)^m (2m+n)x^{2m+n}}{2^{2m+n} m!\,(m+n)!} + \sum_{m=0}^{\infty}m(m-2n)A_m x^{m-n} + \sum_{m=0}^{\infty}A_m x^{m-n+2} = 0 \quad [67]$$

Second Frobenius method solution for n = 0 – At this point we want to return to the separate consideration of n = 0. Recall that we had to set n = 0 and k = 1 in this case. We also had to increase the lower limit on the summation of the $A_m x^m$ terms from zero to one. Making these changes in equation [67] gives the following result for n = 0.

$$2\sum_{m=0}^{\infty}\frac{(-1)^m (2m)x^{2m}}{2^{2m}\left(m!\right)^2} + \sum_{m=1}^{\infty}m^2 A_m x^m + \sum_{m=1}^{\infty}A_m x^{m+2} = 0 \quad [68]$$

We can combine the last two sums if we temporarily replace m by j = m + 2 in the second sum. (Once we get the correct form for this sum, we can replace j by m and combine the two sums. To do this we explicitly write the terms for m = 1 and m = 2 in the first sum.)

$$\sum_{m=1}^{\infty}m^2 A_m x^m + \sum_{m=1}^{\infty}A_m x^{m+2} = \sum_{m=1}^{\infty}m^2 A_m x^m + \sum_{j=3}^{\infty}A_{j-2}x^j = \sum_{m=1}^{\infty}m^2 A_m x^m$$
$$+ \sum_{m=3}^{\infty}A_{m-2}x^m = A_1 x + 4A_2 x^2 + \sum_{m=3}^{\infty}\left[m^2 A_m + A_{m-2}\right]x^m \quad [69]$$

We can use equation [69] to replace the last two sums in equation [68]. In addition, we see that the 2m in the numerator of the first sum in equation [68] makes the first term in that sum zero. Thus, we can change the lower limit of that sum from m = 0 to m = 1. Making these changes allows us to rewrite equation [68] as follows.

$$\sum_{m=1}^{\infty}\frac{(-1)^m (4m)x^{2m}}{2^{2m}\left(m!\right)^2} + A_1 x + 4A_2 x^2 + \sum_{m=3}^{\infty}\left[m^2 A_m + A_{m-2}\right]x^m = 0 \quad [70]$$

In the first sum, we can combine the 4 in the numerator with the 2^{2m} in the denominator and we can also write $m/(m!)^2$ as $1/[m!(m-1)!]$. This gives the following change to equation [70].

$$\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-2}m!(m-1)!}+A_1x+4A_2x^2+\sum_{m=3}^{\infty}\left[m^2A_m+A_{m-2}\right]x^m=0 \qquad [71]$$

Since equation [71] is a power series whose sum is equal to zero, the coefficient of every power in this series must vanish to for the sum to equal zero for any value of x. We see that the lowest power of x occurs in the term A_1x. We must have $A_1 = 0$ for the x coefficient to vanish.

We next consider the x^2 term. For the coefficient of this term to vanish we must satisfy the following equation.

$$\frac{(-1)^1}{2^{2(1)-2}1!(1-1)!}+4A_2=1+4A_2=0 \qquad \Rightarrow \qquad A_2=\frac{1}{4} \qquad [72]$$

Since the first sum in equation [71], which comes from the first derivative of $J_n(x)$, has only even powers of x, the coefficients for odd powers of x are given by the second sum only. Setting the coefficient of odd powers of x equal to zero gives the following result.

$$A_m=\frac{A_{m-2}}{m^2} \qquad odd\ m \qquad [73]$$

Since we have previously shown that $A_1 = 0$, equation [73] tells us that $A_3 = 0$. In fact we can apply it sequentially to all values of A_m with an odd subscript to show that all values of A_m for which m is odd are zero. This leaves us with only even values of m to consider. For convenience we can rewrite the second sum in equation [71] to have only even powers. To do this we temporarily define the index k = m/2 and replace m by 2k in the second sum in equation [71]. Before doing this, we set $A_2 = 0$ and start the sum at m = 4 since the m = 3 term is zero.

$$\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-2}m!(m-1)!}+4A_2x^2+\sum_{k=2}^{\infty}\left[(2k)^2A_{2k}+A_{2k-2}\right]x^{2k}=0 \qquad [74]$$

From equation [74], we can see that the coefficients of even powers, x^{2m}, for m > 1, will vanish if the following equation is satisfied.

$$\frac{(-1)^m}{2^{2m-2}m!(m-1)!}+(2m)^2A_{2m}+A_{2m-2}=0 \qquad m>1 \qquad [75]$$

This allows us to define new coefficients in terms of old ones by the following equation.

$$A_{2m}=-\frac{(-1)^m}{2^{2m-2}(2m)^2m!(m-1)!}-\frac{A_{2m-2}}{(2m)^2} \qquad [76]$$

For m = 2, we can compute A_4 in terms of A_2 = ¼.

$$A_4 = -\frac{(-1)^2}{2^{2(2)-2}(2\cdot 2)^2(2!)(2-1)!} - \frac{A_2}{(2\cdot 2)^2} = -\frac{1}{(4)(16)(2)(1)} - \frac{1/4}{16} = -\frac{3}{128} \qquad [77]$$

For m = 3, we can compute A_6 in terms of A_4 and use the result that A_4 = -3/128 to get a value for A_6.

$$A_6 = -\frac{(-1)^3}{2^{2(3)-2}(2\cdot 3)^2(3!)(3-1)!} - \frac{A_4}{(2\cdot 3)^2} = -\frac{-1}{(16)(36)(6)(2)} - \frac{-3/128}{36}$$
$$= \frac{1}{6{,}912} + \frac{3}{4{,}608} = \frac{2}{13{,}284} + \frac{9}{13{,}284} = \frac{11}{13{,}284} \qquad [78]$$

The general form for A_{2m} is not easy to see. We can see that the recursion equation [76] for A_{2m} contains one term from the series sum for the first derivative of J_0 and a second term which contains the previous value of A_{2m}. Because this continues indefinitely, general expression will look like the following expression that we found for A_6.

$$A_6 = \begin{bmatrix} m=3 \\ J_0^{'}\ term \end{bmatrix} - \frac{A_4}{(2\cdot 3)^2} = \begin{bmatrix} m=3 \\ J_0^{'}\ term \end{bmatrix} - \frac{1}{(2\cdot 3)^2}\left\{\begin{bmatrix} m=2 \\ J_0^{'}\ term \end{bmatrix} - \frac{A_2}{(2\cdot 2)^2}\right\}$$
$$= \begin{bmatrix} m=3 \\ J_0^{'}\ term \end{bmatrix} - \frac{1}{(2\cdot 3)^2}\left\{\begin{bmatrix} m=2 \\ J_0^{'}\ term \end{bmatrix} - \frac{1}{(2\cdot 2)^2}\left(\begin{bmatrix} m=1 \\ J_0^{'}\ term \end{bmatrix}\right)\right\} \qquad [79]$$

Equation [79] shows the general form that the $J_0^{'}$ term appears repeatedly with a series of factors. Although we will not attempt to derive the general equation below, equation [79] tells us that the form of the equation proposed in [80] for A_{2m} is reasonable.

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k} \qquad [80]$$

We can show that this result satisfies the difference equation in [76] as follows. First, we rewrite equation [80], replacing m by m – 1.

$$A_{2m-2} = \frac{(-1)^{m-2}}{2^{2m-2}\left[(m-1)!\right]^2}\sum_{k=1}^{m-1}\frac{1}{k} \qquad [81]$$

We then substitute equation [81] into equation [76] and after some manipulation we see that we obtain the same result for A_{2m} given by equation [80].

$$A_{2m} = -\frac{(-1)^m}{2^{2m-2}(2m)^2 m!(m-1)!} - \frac{1}{(2m)^2}\frac{(-1)^{m-2}}{2^{2m-2}[(m-1)!]^2}\sum_{k=1}^{m-1}\frac{1}{k}$$
$$= \frac{(-1)^{m-1}}{2^{2m-2}2^2 m^2 m(m-1)!(m-1)!} + \frac{(-1)^{m-1}}{2^{2m-2}2^2 m^2[(m-1)!]^2}\sum_{k=1}^{m-1}\frac{1}{k}$$
$$= \frac{(-1)^{m-1}}{2^{2m} m[m(m-1)!][m(m-1)!]} + \frac{(-1)^{m-1}}{2^{2m}[m(m-1)!]^2}\sum_{k=1}^{m-1}\frac{1}{k} \quad [82]$$
$$= \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left[\frac{1}{m} + \sum_{k=1}^{m-1}\frac{1}{k}\right] = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}$$

Since the last expression in equation [82] is the general expression proposed for A_{2m} in equation [80], we conclude that equation [80] is a correct solution to the recusrion equation that we derived for A_{2m} in terms of A_{2m-2} in equation [76].

If we substitute this equation [80] for A_{2m} into the original equation for our second solution in [58], we get the result that

$$y_2(x) = J_0(x)\ln(x) + \sum_{m=1}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}\right\} \quad [83]$$

Although this solution provides a second linearly independent solution to Bessel's equation for ν = 0, it is conventional to tabulate the function $Y_0(x)$ shown below as the second solution. In this equation γ is called the Euler constant. It has a value of 0.5772156649015328606061 52…, and represents the limit of the following sum as x approaches infinity: 1 + ½ + 1/3 + ¼ +⋯+1/x – ln x.

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2)J_0(x)\right] = \frac{2}{\pi}\left[J_0(x)\ln(x) + \sum_{m=1}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}\right\}\right.$$
$$\left. + (\gamma - \ln 2)J_0(x)\right] = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}\right\}\right] \quad [84]$$

Second Frobenius method solution for integer $\nu \neq 0$ – Having completed the second solution for J_0, we can return to the general case of finding the second solution for J_n, where n is nonzero. We return to equation [67], the last equation in our general development for any integer ν = n, and rewrite the final two sums in that equation as follows.

$$\sum_{m=0}^{\infty} m(m-2n)A_m x^{m-n} + \sum_{m=0}^{\infty} A_m x^{m-n+2} = \sum_{m=0}^{\infty} m(m-2n)A_m x^{m-n}$$
$$+ \sum_{j=2}^{\infty} A_{j-2}x^{j-n} = 0 + (1-2n)A_1 x^{1-n} + \sum_{j=2}^{\infty}\left[j(j-2n)A_j + A_{j-2}\right]x^{j-n} \quad [85]$$

Substituting this equation into equation [67] gives the following result.

$$2k\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n}}{2^{2m+n}m!\,(m+n)!} + (1-2n)A_1 x^{1-n} + \sum_{j=2}^{\infty}\left[j(j-2n)A_j + A_{j-2}\right]x^{j-n} = 0 \quad [86]$$

The minimum power of x in this equation is x^{1-n}. The coefficient of this term is (1 – 2n) A_1. Since n is an integer, this coefficient will only vanish if $A_1 = 0$.

Because the first term has x raised to the 2m+n power while the other terms have x raised to the j–n power, the two different summations will have common powers of x only when 2m + n = j – n. That is when j = 2m + 2n. Since the minimum value of m is zero, the first sum will only come into play when j ≥ 2n. For j < 2n, then, the coefficients of the x terms will vanish only if the coefficients of x^{j-n} in the second summation are zero. This requires

$$j(j-2n)A_j + A_{j-2} = 0 \qquad \Rightarrow \qquad A_j = \frac{A_{j-2}}{j(j-2n)} \qquad [87]$$

Since $A_1 = 0$, this equation tells us that all values of A_j, with an odd number for j, with j < 2n are zero. For even j < 2n, the coefficients are be expressed in terms of A_0, which will remain an unknown. For j = 2, 4, and 6, we have the following results from equation [87].

$$A_2 = \frac{A_0}{2(2-2n)} \qquad A_4 = \frac{A_2}{4(4-2n)} = \frac{\left(\frac{A_0}{2(2-2n)}\right)}{4(4-2n)} = \frac{A_0}{2\cdot 4(2-2n)(4-2n)}$$
$$A_6 = \frac{A_6 4}{6(6-2n)} = \frac{\left(\frac{A_0}{2\cdot 4(2-2n)(4-2n)}\right)}{4(4-2n)} = \frac{A_0}{2\cdot 4\cdot 6(2-2n)(4-2n)(6-2n)} \qquad [88]$$

We can continue in this fashion until we compute A_{2n-2}; the next value of j will be j = 2n. At this point, the exponent of x^{j-n} in the second sum of equation [86] becomes equal to 2n – n = n. This is the same power of x that occurs at the lower limit of m = 0 in the first sum of equation [86]. So for j ≥ n, we have to consider both sums in equation [86]. The following equation, obtained by setting m = 0 in the first sum and j = 2n in the second sum, is required to set the coefficient of x^n in equation [86] equal to zero.

$$2k\frac{(-1)^0(2\cdot 0+n)}{2^{2\cdot 0+n}0!\,(0+n)!} + 2n(2n-2n)A_{2n} + A_{2n-2} = 0 \qquad [89]$$

We can solve this equation for k, the coefficient of the $J_n(x)\ln(x)$ term in the proposed second solution

$$k = \frac{2^n n!\,A_{2n-2}}{2n} = \kappa A_0 \qquad [90]$$

We will develop an equation for k later in the derivation. Note that the value of k may be zero in other solutions by Frobenius method where the roots of the indicial equation differ by an integer.

The remaining powers of x in the series have contributions from both sums. The power of x in the two sums will be the same if j = 2m + 2n. (This covers only even powers of x, which are the only powers left in the series whose coefficients are not already zero.) Setting j = 2m + 2n in the final sum of equation [86] gives the following equation to make the coefficients x^{2m+n} vanish.

$$2k\frac{(-1)^m(2m+n)}{2^{2m+n}m!\,(m+n)!}+(2m+2n)(2m)A_{2m+2n}+A_{2m+2n-2}=0 \qquad [91]$$

If we set m = 0 in this equation we recover equation [89] which we used to solve for k. Thus, this equation only applies for m ≥ 1. We can solve this equation for A_{2m+2n} as follows

$$A_{2m+2n}=-2k\frac{(-1)^m(2m+n)}{2^{2m+n}(2m+2n)(2m)m!\,(m+n)!}-\frac{A_{2m+2n-2}}{(2m+2n)(2m)} \qquad m\geq 1 \qquad [92]$$

When we apply this equation for m = 1, using the result that k = k A_0, we obtain A_{2n+2},

$$A_{2n+2}=-2\kappa A_0\frac{(-1)^1(2+n)}{2^{2+n}(2+2n)(2)1!\,(1+n)!}-\frac{A_{2n}}{(2+2n)(2)}=\frac{\kappa A_0(2+n)}{2^{3+n}(1+n)\,(1+n)!}-\frac{A_{2n}}{(4+4n)}$$

[93]

We can continue to apply equation [94] to get additional values of A_m. The general result for this coefficient is tedious to obtain and we will skip the steps that lead to the following result.

$$y_2(x)=J_n(x)\ln(x)+x^n\sum_{m=0}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m+n+2}m!(m+n)!}\left(\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{m+n}\frac{1}{k}\right)\right\}+x^{-n}\sum_{m=0}^{n-1}\left\{\frac{(n-m-1)!\,x^{2m}}{2^{2m-n+2}m!(m+n)!}\right\}$$

[94]

Although this solution provides a second linearly independent solution to Bessel's equation for ν = n ≠ 0, it is conventional to tabulate the function $Y_n(x)$ shown below as the second solution. The definition of this second solution is similar to the one used in obtaining equation [84] for $Y_0(x)$.

$$Y_n(x)=\frac{2}{\pi}\left[y_2(x)+(\gamma-\ln 2)J_n(x)\right]=$$

$$\frac{2}{\pi}\left[J_n(x)\left(\ln\frac{x}{2}+\gamma\right)+x^n\sum_{m=0}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m+n+2}m!(m+n)!}\left(\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{m+n}\frac{1}{k}\right)\right\}+x^{-n}\sum_{m=0}^{n-1}\left\{\frac{(n-m-1)!\,x^{2m}}{2^{2m-n+2}m!(m+n)!}\right\}\right]$$

[95]

If we set n = 0 in equation [95], we will obtain equation [84] for $Y_0(x)$ if the final sum in equation [95] is omitted when n = 0.

There is one final step that is taken to obtain consistency between integer-order and noninteger-order Bessel functions. Although J_ν and $J_{-\nu}$ are linearly independent solutions for noninteger ν, we define an alternative second solution for noninteger n by the following equation.

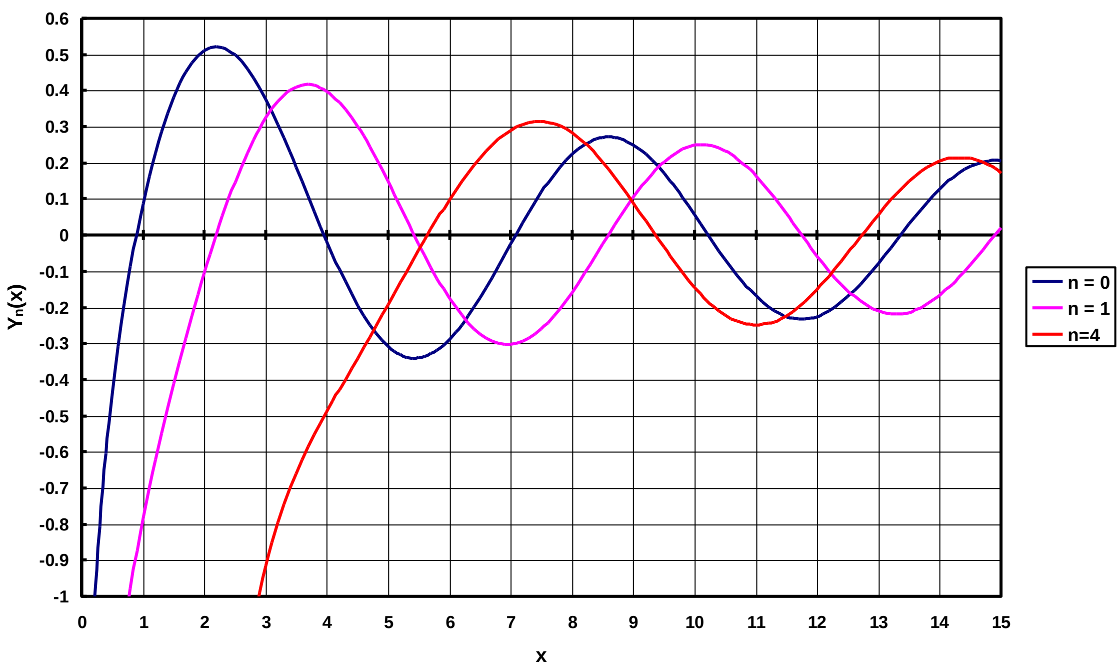
$$Y_\nu(x)=\frac{(\cos\nu\pi)J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi} \qquad [96]$$

In the limit as ν approaches an integer, the numerator and denominator of equation [96] both approach zero. (Recall equation [57] for Bessel functions of integer order: $J_{-n}(x) = (-1)^nJ_n(x)$; since cos nπ = $(-1)^n$, the numerator approaches zero as ν approaches an integer.) If we apply L'Hopital's rule to equation [96] as ν approaches integer n, we can show that the result we get is the same as that given for $Y_n(x)$ in equation [95].

The basic result of this section on Bessel functions then is that the solution to Bessel's equation [36] is $y = C_1J_\nu(x) + C_2Y_\nu(x)$ for both integer and noninteger ν. Here C_1 and C_2 are constants that are determined by the boundary conditions on the differential equation. In the same way that we recognize that $y = C_1\sin(kx) + C_2\cos(kx)$ is the solution to $d^2y/dx^2 + k^2y = 0$, we can now recognize that $y = C_1J_\nu(x) + C_2Y_\nu(x)$ is the solution to $x^2d^2y/dx^2 + xdy/dx + (x^2 - \nu^2)y = 0$. Of course, Bessel function tables are not as common as tables of sines and cosines and you probably do not have a Bessel function button on your calculator. However, on an Excel spreadsheet you can get Bessel functions $J_\nu(x)$, for both integer and noninteger ν, by the function besselj(x,ν). Similarly, you can get $Y_\nu(x)$, for both integer and noninteger ν, by the function bessely(x,ν). These functions were used to prepare the plots of $J_n(x)$ shown above the the plots of $Y_n(x)$ shown below.

This plot shows that the values of $Y_n(x)$ approach minus infinity as x approaches zero due to the ln(x) term in $Y_n(x)$. Because the common application of Bessel functions is to problems with radial geometries, we usually have to take $C_2 = 0$ in the general solution, $y = C_1J_\nu(x) + C_2Y_\nu(x)$, to have a solution that remains finite at x = 0.

Bessel Functions of the Second Kind of Integer Order



Summary of Frobenius method

Frobenius method is used to determine series solutions to differential equations with the following form: $x^2d^2y/dx^2 + xb(x)dy/dx + c(x)y = 0$. In these notes, we have applied this method to the solution of Bessel's equation.

- The general form of the Frobenius method solution is the infinite series $y(x) = x^r(a_0 + a_1x + a_2x^2 + \ldots)$

- The general solution is differentiated and substituted into the original differential equation. Setting the coefficients of each power of x^n equal to zero gives equations that can be solved for r and the a_i coefficients.

- Setting the coefficient of $x^r = 0$ gives a quadratic equation for r known as the indicial equation.

- There are three possible cases for the two roots of the indicial equation: (1) the two roots are the same; (2) the two roots differ by an integer (other than zero); (3) the two roots are different and their difference is not an integer.

- In all three cases, the first solution is $y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \ldots.)$, where r is one root to the indicial equation. This must be taken as the greater of the two roots if the roots differ by an integer.

- The values of the a_i coefficients in the solution are determined in the same way as in the general power series solution; the coefficients of each power of x^n in the solution must be zero.

- If the two roots of the indicial equation are different, and the difference is not an integer, the second solution is is $y_2(x) = x^R(A_0 + A_1x + A_2x^2 + \ldots.)$, where R is the second root to the indicial equation.

- If the two roots are the same, the second solution is is $y_2(x) = y_1(x) \ln(x) + (A_1x + A_2x^2 + A_3x^3 + \ldots.)$.

- If the two roots differ by an integer, the second solution is is $y_2(x) = k\, y_1(x) \ln(x) + (A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots.)$, where k may be zero.

- The coefficients A_i (and the value of k) are found by making the coefficients of all powers of x equal to zero.

You should understand how Frobenius method works, and be able to apply this method. However, it does not play a role in the solution of most practical engineering problems.

The Sturm-Liouville problem

The Sturm-Liouville problem forms the basis for several problems in engineering analysis. It provides a link among different functions including Bessel functions, sines and cosines, and other special functions. Sets of these functions, which are solutions to the Sturm-Liouville problem, have a common ability to represent ***any*** function over a certain region in space.

The Sturm-Liouville problem is defined by the following differential equation over a region $a \le x \le b$.

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + \left[q(x) + \lambda p(x)\right]y = 0 \qquad [97]$$

with the following boundary conditions at x = a and x = b. In these boundary conditions, at least one of the two constants k_1 and k_2 is not zero. Similarly, at least one of the two constants ℓ_1 and ℓ_2 is not zero.

$$k_1 y(a) + k_2 \left.\frac{dy}{dx}\right|_{x=a} = 0 \qquad and \qquad \ell_1 y(b) + \ell_2 \left.\frac{dy}{dx}\right|_{x=b} = 0 \qquad [98]$$

The functions r(x), dr(x)/dr, q(x) and p(x) must be continuous in the region a ≤ x ≤ b and we must have p(x) > 0. The Sturm-Liouville problem is one of a general class of problems involving linear operators. We can define a general linear operator, L, which may be a derivative operator such as d/dx or d^2/dx^2, or a combination of operations such as (d^2/dx^2 + 1). If our linear operator, L, operates on a function and returns the same function times a constant, λ, we call the function an eigenfunction of the operator and the constant, λ, an eigenvalue.

If Lf(x) = λf(x), then f(x) is an eigenfunction of L and λ is an eigenvalue [99]

Note the similarity to the matrix eigenvalue problem, **Ax** = λ**x**. For example, if L is the simple first derivative operator, equation [99] becomes

$$\frac{df}{dx} = \lambda f \qquad \Rightarrow \qquad f = e^{\lambda x} \qquad [100]$$

We say that $e^{\lambda x}$ is the eigenfunction of the operator d/dx. We can define a generalized eigenfunction problem by the following equation.

Lf(x) = λp(x)f(x) [101]

Here p(x) is called the weight function. We see that the Sturm-Liouville problem is an eigenfunction problem of with the form of equation [101]; the definition of the operator for this problem can be found by examining equation [97].

$$L = -\frac{d^2}{dx^2} - r(x)\frac{d}{dx} - q(x) \qquad [102]$$

For functions, such as those in the solutions to the Sturm-Liouville problem, the inner product is defined as follows.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x) f_j(x) p(x) dx \qquad [103]$$

In this definition, the function p(x), which appears in the Sturm-Liouville problem is called the weighting function. In many cases, p(x) =1 and is not considered in the definition of the inner product of functions. For the Sturm-Liouville problem, p(x) is defined to be always greater than zero.

The Strum-Liouville operator is one of a class of operators known as Hermetian or self-adjoint operators. The linear operator, L, has an adjoint operator, L*, which is also linear, defined by the following inner product equation. Here f_i and f_j are functions on which the operators L and L* act.

$$(Lf_i, f_j) = (f_i, L^* f_j) \qquad [104]$$

It is possible to have an operator that is self-adjoint or Hermetian operator; that is an operator for which L* = L. We can show that the operator in the Sturm-Liouville equation, as defined in equation [102], satisfies the definition of a self-adjoint using the inner-product definition in equation [103]. For a self-adjoint operator, equation [104] tells us that $(Lf_i,f_j) = (f_i,Lf_j)$.

$$\int_a^b \left(-\frac{d^2 f_i}{dx^2} - r(x)\frac{df_i}{dx} - q(x)\right) f_j p(x)dx = \int_a^b f_i \left(-\frac{d^2 f_j}{dx^2} - r(x)\frac{df_j}{dx} - q(x)\right) fp(x)dx$$

[105]

We now state two important results for Hermetian (or self-adjoint) operators in general and for the Sturm-Liouville operator in particular:

1. The eigenvalues of any self-adjoint or Hermetian operator are real.

2. The eigenfunctions of any self-adjoint or Hermetian operator defined over a region $a \le x \le b$ form an orthogonal set over that region.

3. The eigenfunctions form a complete set if the vector space has a fininte number of dimensions as in an n x n matrix.

4. The Sturm-Liouville operator has a complete set of eigenfunctions over an infinite-dimensional vector space.

The notion of orthogonal functions is an extension of the notions of inner products and orthogonality for vectors. For functions, the inner product is defined in terms of the integral in equation [103]. If we have a set of functions, $f_i(x)$ defined on an interval $a \le x \le b$, are orthogonal over that interval if the following relationship holds.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x) f_j(x) p(x)dx = a_i\delta_{ij} \qquad [106]$$

We can define a set of ***orthonormal*** functions by the following equation.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x) f_j(x) p(x)dx = \delta_{ij} \qquad [107]$$

Any set of orthogonal functions $g_i(x)$ can be converted to a set of orthonormal functions $f_i(x)$ by dividing by the square root of the inner product, (g_i,g_i), which we can also write as the two norm, $||g_i||_2$. If we understand that we are always using the two-norm, we can drop the subscript and write this more simply as $||g_i||$.

$$f_i(x) = \frac{g_i(x)}{\|g_i\|} = \frac{g_i(x)}{\sqrt{(g_i, g_i)}} = \frac{g_i(x)}{\sqrt{\int_a^b g_i^*(x) g_j(x) p(x)dx}} \qquad [108]$$

Eigenfunction expansions

If we have a complete set of eigenfunctions in a region $a \le x \le b$ we can expand any other function, f(x), in that region by the following equation.

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) \quad [109]$$

We can derive a simple expression for the coefficients in this equation if the eigenfunctions are orthogonal. If we multiply both sides of this equation by $\rho(x)y_n(x)$ and integrate between x = a and x = b, we obtain the following result.

$$\int_a^b p(x) y_n(x) f(x) dx = \int_a^b p(x) y_n(x) \sum_{m=0}^{\infty} a_m y_m(x) dx = \sum_{m=0}^{\infty} a_m \int_a^b p(x) y_n(x) y_m(x) dx \quad [110]$$

We can rewrite these integrals using the inner product notation.

$$(y_n, f) = \sum_{m=0}^{\infty} a_m (y_n, y_m) \quad [111]$$

Since the y_m form an orthogonal set, the only nonzero term in the summation on the right is the one for which n = m. All other inner products are zero because of orthgonality. Thus we have a simple equation to solve for a_m after setting n = m everywhere in the equation.

$$a_m = \frac{(y_m, f)}{(y_m, y_m)} = \frac{(y_m, f)}{\|y_m\|^2} = \frac{\int_a^b p(x) y_m(x) f(x) dx}{\int_a^b p(x) y_m(x) y_m(x) dx} \quad [112]$$

If we have an orthonormal set, the denominator of the a_m equation is one.

The most common eigenfunction expansions are the Fourier series of sines and cosines. These functions are solutions of a Sturm Liouville differential equation defined by equations [97] and [98]. To see this we consider the differential equation $d^2y/dx^2 + \omega^2 y$ defined over $0 \le x \le 1$, with y(0) = 0 and y(1) = 0. This is a Sturm-Liouville equation with r(x) = p(x) = 1 , $\lambda = \omega^2$, and q(x) = 0. The boundary conditions satisfy equation [98] with $k_1 = \ell_1 = 1$and $k_2 = \ell_2 = 0$. The set of functions $y_m = \sin(\omega_m x) = \sin(m\pi x)$, where m is any integer, satisfied both the differential equation and the boundary conditions. (You can show this by substitution into the differential equation. You should be able to say why the cosine will not be a solution to this problem.)

The solutions are orthogonal, but we cannot tell if they are orthonormal unless find the norm of a function. This requires the following integral for the norm. (Note that p(x) was equal to one in our problem definition so it does not appear in the inner product.) The resulting integral is evaluated using integral tables for indefinite integral of $\sin^2(ax)$.

$$\|y_m\|^2 = (y_m, y_m) = \int_0^1 \sin^2(m\pi x) dx = \frac{1}{m\pi}\left[\frac{m\pi x}{2} - \frac{\sin(2m\pi x)}{4}\right]_0^1 = \frac{1}{2} \quad [113]$$

Thus, for this set of functions, the coefficients in an eigenfunction expansion are computed from the following modification of equation [112].

$$a_m = \frac{(y_m, f)}{\|y_m\|} = \frac{(y_m, f)}{\left(\frac{1}{2}\right)} = 2\int_0^1 \sin(m\pi x)\, f(x)dx \qquad [114]$$

For example, we can expand the simple function f(x) = c, a constant, using the following coefficients.

$$a_m = 2\int_a^b \sin(m\pi x)cdx = \frac{2c}{m\pi}\left[-\cos(m\pi x)\right]_0^1 = \frac{2c}{m\pi}\left[1-\cos(m\pi)\right] \qquad [115]$$

The term 1 – cos(mπ) = 0 if m is even and 2 if m is odd. We can thus write the a_m coefficients for this expansion as follows.

$$a_m = \begin{cases} 0 & m\ even \\ \dfrac{4c}{m\pi} & m\ odd \end{cases} \qquad [116]$$

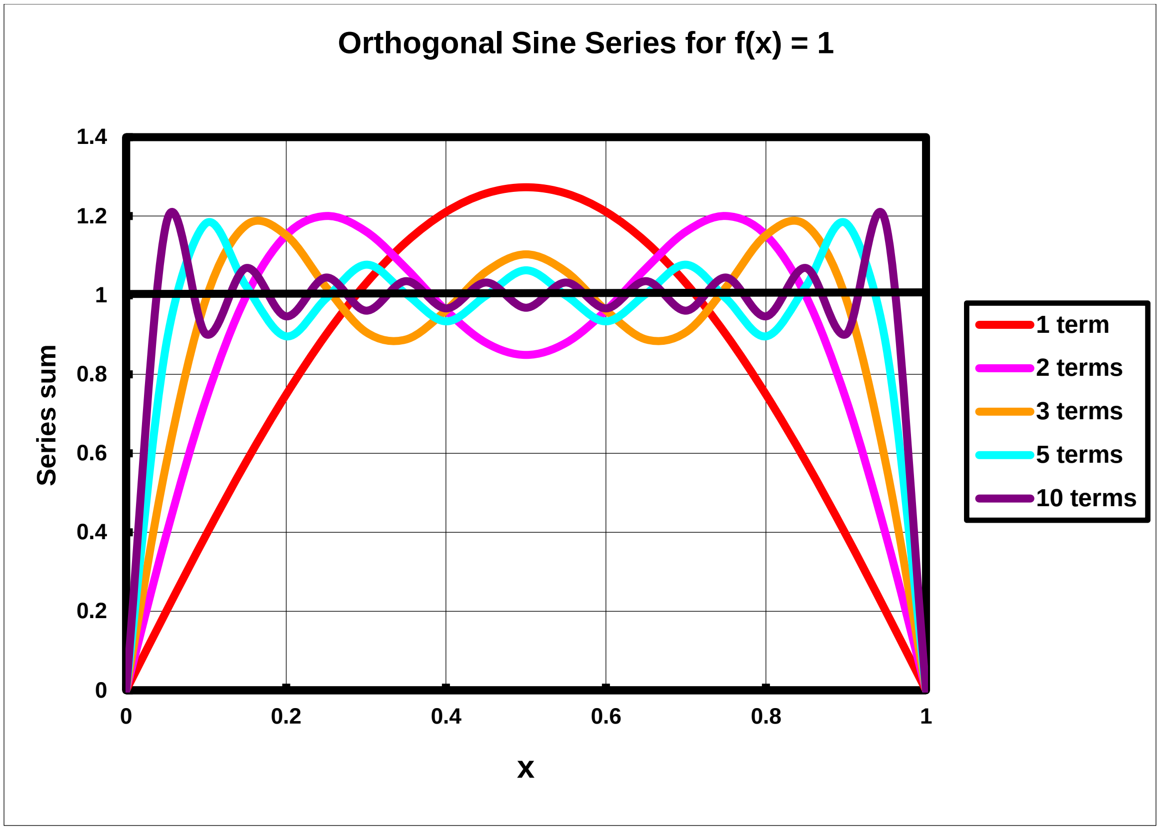
In this case, our eigenfunction expansion in equation [109] becomes

$$f(x) = c = \sum_{m=0}^{\infty} a_m y_m(x) = \sum_{m=1,3,5}^{\infty} \frac{4c}{m\pi}\sin(m\pi x) \qquad [117]$$

We see that we can divide by c and obtain the following expression.

$$1 = \frac{4}{\pi}\sum_{m=1,3,5,\dots}^{\infty} \frac{\sin(m\pi x)}{m} \qquad [118]$$

The partial sums of this series for a small number of terms are shown in the figure below.



We see that, as we take additional terms, the series converges to the value of f(x) = 1, but there are continued oscillations about this point. In addition, at the boundaries of x = 0 and x = 1, the series sum is zero.

Bessel's equation as a Sturm-Liouville problem and its eigenfunction expansions

We can show that Bessel's equation [36] is a Sturm-Liouville equation. We multiply equation [36] by x^2 and define a new variable, z = x/k such that dy/dz = (1/k)dy/dz and $(1/k^2)d^2y/dz^2$. We can the rewrite equation [36] in terms of z. If we divide the result by z, we obtain.

$$\frac{1}{z}\left[(kz)^2\frac{1}{k^2}\frac{d^2y}{dz^2}+kz\frac{1}{k}\frac{dy}{dz}+\left(k^2z^2-\nu^2\right)y\right]=z\frac{d^2y}{dz^2}+\frac{dy}{dz}+\left(-\frac{\nu^2}{z}+k^2z\right)y=0 \quad [119]$$

However, since we started with an equation for y as a function of x in equation [36] and we have not changed this, we could formally write y = y(x) = y(kz) in equation [119] and below in equation [120]. Since zd^2y/dz^2 + dy/dz can be written[4] as d[z dy/dz]/dz, we can reduce equation [119] to the Sturm Liouville form of equation [97] as follows.

$$z\frac{d^2y}{dz^2}+\frac{dy}{dz}+\left(-\frac{\nu^2}{z}+k^2z\right)y=\frac{d}{dz}\left(z\frac{dy}{dz}\right)+\left(-\frac{\nu^2}{z}+k^2z\right)y=\frac{d}{dz}\left(r\frac{dy}{dz}\right)+\left(q+\lambda z\right)y=0$$

[120]

[4] You can show that this is correct by applying the rule for differentiation of products to d[z dy/dz]/dz.

We see that Bessel's equation has the following definitions of the functions r, q, and p, and the eigenvalue λ, in the Sturm-Liouville equation: r = z, q = $-n^2/z$, p = z, and $\lambda = k^2$.

If we consider the case of integer ν = n over a region $0 \le z \le R$ the general solution of Bessel's equation, remembering that we now have y(kz) as our dependent variable, is a linear combination of $J_n(kz)$ and $Y_n(kz)$. However, $Y_n(kz)$ approaches infinity as z approaches zero. Thus, $Y_n(kz)$ cannot be part of our general solution when we include z = 0. If the boundary condition at z = R is y(kR) = 0, we must select k to satisfy this equation. It turns out that there are an infinite number of values that we can select for which kR = 0. There are known as the zeros of the Bessel function. (See the plots of the Bessel function above that show the initial locations at which the first few Bessel functions are zero.)

For convenience, we define α_{mn} and k_{mn} as follows.

$$J_n(\alpha_{mn}) = J_n(k_{mn}R) = 0 \qquad k_{mn} = \frac{\alpha_{mn}}{R} \tag{121}$$

The values of α_{mn}, called the zeros of J_n, are the values of the argument of J_n for which J_n is zero. There are an infinite of values, α_{mn}, for which $J_n(\alpha_{mn}) = 0$. Note the two subscripts for α_{mn}; m is an eigenfunction-counting index that ranges from one to infinity. Do not confuse this eigenfunction index with the index, n, for J_n. The index for J_n is set by the appearance of n in the original differential equation. Thus our set of eigenfunctions, for fixed n, will be $J_n(k_{1n}z)$, $J_n(k_{2n}z)$, $J_n(k_{3n}z)$, $J_n(k_{4n}z)$, etc.

This set of functions, $J_n(k_{mn}z)$ is orthogonal, since it is a solution to a Sturm-Liouville problem. The orthogonality condition satisfied by these functions is defined from equation [106] with a weighting function, p(z) = z, as discussed in the paragraph following equation [120]. Applying equation [106] to this problem we have the interval from 0 to R (in place of a to b) and we have a real function so we do not have to consider the complex conjugate. This gives the following orthogonality condition.

$$(f_m, f_o) = (J_n(k_{mnz}z), J_n(k_{onz}r)) = \int_0^R J_n(k_{mnz}z) J_n(k_{onz}z) z dz = a_m \delta_{mo} \tag{122}$$

We can apply the usual equation for an eigenfunction expansion from [109] to this set of Bessel functions.

$$f(z) = \sum_{m=1}^{\infty} a_m y_m(z) = \sum_{m=1}^{\infty} a_m J_n(k_{mn}z) \tag{123}$$

If we multiply both sides of this equation by $zJ_n(k_{on}z)dz$ and integrate from 0 to R we obtain the following result.

$$\int_0^R f(z) J_n(k_{on}z) z dz = \sum_{m=1}^{\infty} a_m \int_0^R J_n(k_{mn}z) J_n(k_{on}z) z dz = a_o \int_0^R J_o^2(k_{on}z) z dz \tag{123}$$

In the final step we use the orthogonality relationship which makes all terms in the sum, except for the term in which m = o, zero. We can use an integral table to evaluate the normalization integral.[5]

$$\int_0^R J_o^2(k_{on}z)zdz = \frac{R^2 J_1^2(k_{on}z)}{2} \qquad [124]$$

Combining equations [123] and [124] and using m in place of o as the coefficient subscript gives the following result for the eigenfunction expansion coefficients.

$$a_m = \frac{2}{R^2 J_1^2(k_{mn}z)} \int_0^R f(z) J_n(k_{mn}z)zdz \qquad [123]$$

Although this seems like an unlikely choice of eigenfunctions with which to expand an arbitrary f(x), we will see that such expansions become important in the consideration of partial differential equations in cylindrical geometries.

Summary of these notes

We have developed the background to solve the Sturm-Liouville problem from several general cases by developing the tools of power series solutions and Frobenius method. We showed the details of how one applies Frobenius method to the solution of Bessel's equation. The Kreyszig text covers other applications of this method.

We discussed the Sturm-Liouville problem. We noted that this problem was defined by the differential equation and boundary conditions in equations [97] and [98] that are copied below.

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + [q(x) + \lambda p(x)]y = 0 \qquad [97]$$

In the boundary conditions below, at least one of the two constants k_1 and k_2 is not zero. Similarly, at least one of the two constants ℓ_1 and ℓ_2 is not zero. It is also possible to have periodic boundary conditions. These are discussed by Kreyszig.

$$k_1 y(a) + k_2 \left.\frac{dy}{dx}\right|_{x=a} = 0 \qquad and \qquad \ell_1 y(b) + \ell_2 \left.\frac{dy}{dx}\right|_{x=b} = 0 \qquad [98]$$

We noted that the Sturm-Liouville problem was an example of a Hermetian or self-adjoint operator. This operator for functions had properties similar to a Hermetian matrix, which could also be regarded as a Hermetian operator. We listed the following important results for Hermetian (or self-adjoint) operators in general and for the Sturm-Liouville operator in particular:

[5] The general form of this integral, which is found by integration by parts followed by substitution of Bessel's equation, is shown below.

$$\int_a^b J_\nu(\lambda x)xdx = \frac{1}{2}\left[\left(x^2 - \frac{\nu^2}{\lambda^2}\right)J_\nu^2(\lambda x) + x^2\left(\frac{dJ_\nu(\lambda x)}{d(\lambda x)}\right)^2\right]_{x=a}^{x=b}$$

1. The eigenvalues of any self-adjoint or Hermetian operator are real.

2. The eigenfunctions of any self-adjoint or Hermetian operator defined over a region a ≤ x ≤ b form an orthogonal set over that region.

3. The eigenfunctions form a complete set if the vector space has a finite number of dimensions as in an n x n matrix.

4. The Sturm-Liouville operator has a complete set of eigenfunctions over an infinite-dimensional vector space.

For functions, the inner product is defined in terms of the integral in equation [103]. This definition includes a weight function, p(x), which may be 1. The orthogonality condition for functions was given in equation [106], which is copied below.

$$\left(f_i, f_j\right) = \int_a^b f_i^*(x) f_j(x) p(x) dx = a_i \delta_{ij} \qquad [106]$$

If we had an orthonormal set of functions, then the value of a_i in the previous equation would be one.

Because the eigenfunctions of the Sturm-Liouville problem form a complete orthogonal set, we can expand any function in the region in which the Sturm-Liouville problem is defined in terms of these eigenfunctions. Equation [109] gives this general eigenfunction expansion by the following equation $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$, where the coefficients a_m are given by equation [112] copied below.

$$a_m = \frac{\left(y_m, f\right)}{\left(y_m, y_m\right)} = \frac{\left(y_m, f\right)}{\left\|y_m\right\|^2} = \frac{\int_a^b p(x) y_m(x) f(x) dx}{\int_a^b p(x) y_m(x) y_m(x) dx} \qquad [112]$$

Appendix

The various sections in this appendix contain background material on the topics covered in these notes.

Appendix A – Power Series (Taylor Series)

Power series or Taylor series are important tools both in theoretical analysis and in numerical analysis. Recall that the Taylor series for a function of one variable, f(x), expanded about some point, x = a, is given by the infinite series,

$$f(x) = f(a) \quad + \left.\frac{df}{dx}\right|_{x=a}(x-a) \quad + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a}(x-a)^2 \quad + \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=a}(x-a)^3 + \dots \qquad \text{[A-1]}$$

The "x = a" subscript on the derivatives reinforces the fact that these derivatives are evaluated at the expansion point, x = a. We can write the infinite series using a summation notation as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n \qquad \text{[A-2]}$$

In the equation above, we use the definitions of 0! = 1! = 1 and the definition of the zeroth derivative as the function itself. *I.e.*, $d^0f/dx^0|_{x=a} = f(a)$.

If the series is truncated after some finite number of terms, say m terms, the omitted terms are called the **remainder** in mathematical analysis and the **truncation error** in numerical analysis. These omitted terms are also an infinite series. This is illustrated below.

$$\underbrace{f(x) = \sum_{n=0}^{m} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n}_{Terms\ used} + \underbrace{\sum_{n=m+1}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n}_{Truncation\ error} \qquad \text{[A-3]}$$

In this equation, the second sum represents the remainder or truncation error, ε_m, when only m terms are used in the partial sum for the series.

$$\varepsilon_m = \sum_{n=m+1}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n \qquad \text{[A-4]}$$

The theorem of the mean can be used to show that the infinite-series truncation error can be expressed in terms of the first term in the truncation error, that is

$$\varepsilon_m = \frac{1}{(m+1)!} \left.\frac{d^{m+1} f}{dx^{m+1}}\right|_{x=\xi} (x-a)^{m+1} \qquad \text{[A-5]}$$

Here the subscript, "x = ξ", on the derivative indicates that this derivative is no longer evaluated at the known point x = a, but is to be evaluated at x = ξ, an unknown point between x and a. Thus, the price we pay for reducing the infinite series for the truncation error to a single term is that we lose the certainty about the point where the derivative is evaluated. In principle, this would allow us to compute a bound on the error by finding the value of ξ, between x and a, that made the error computed by equation [A-5] a maximum. In practice, we do not usually know the exact functional form, f(x), let alone its $(m+1)^{th}$ derivative. The main result provided by equation [A-5] for use in numerical analysis is that the remainder (or truncation error) depends on the step size, x – a, raised to the power m+1, where m is the number of terms in the partial sum.

In the above discussion, we have not considered whether the series will actually represent the function. A series is said to be **convergent** in a certain region, a ± R, called the **convergence interval**, if we can find a certain number of terms, m, such that the remainder is less than any positive value of a small quantity ε. The half-width of the region, R, is called the **radius of convergence**. A real function, f(x), is called **analytic**, at a point x = a, if it can be represented by a power series in x – a, with a radius of convergence, R > 0.

For simplicity, we can represent the coefficient in a Taylor series as a single coefficient, b_n. This definition and the resulting form for the power series are shown below.

$$b_n = \frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=a} \qquad \textit{so that} \qquad f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n \tag{A-6}$$

We have the following results for operations on power series. Consider two series, $f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$. The sum of the two series is

$$f(x) + g(x) = \sum_{n=0}^{\infty} (b_n + c_n)(x-a)^n \tag{A-7}$$

The product of the two series requires a double summation.

$$\begin{aligned} f(x)g(x) &= \left(\sum_{n=0}^{\infty} b_n (x-a)^n\right)\left(\sum_{m=0}^{\infty} c_m (x-a)^m\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_n (x-a)^n c_m (x-a)^m \\ &= b_0c_0 + (b_0c_1 + b_1c_0)(x-a) + (b_0c_2 + b_1c_1 + b_2c_0)(x-a)^2 + \cdots \end{aligned} \tag{A-8}$$

We can differentiate a series term by term.

$$\frac{df(x)}{dx} = \sum_{n=0}^{\infty} n b_n (x-a)^{n-1} \qquad \frac{d^2 f(x)}{dx^2} = \sum_{n=0}^{\infty} n(n-1) b_n (x-a)^{n-2} \tag{A-9}$$

Appendix B – Factorials

In the discussion above, we have introduced the concept of the factorial of a number. This is defined as follows.

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

From this definition, we can see that 2! = (2)(1) = 2; 3! = (3)(2)(1) = 6; 4! = (4)(3)(2)(1) = 24. We see that higher factorials can be computed in terms of smaller factorials. The examples above show that 4! = 4(3!) and 3! = 3(2!). In general, we can write that

$$n! = n(n-1)! \qquad or \qquad (n-1)! = \frac{n!}{n}$$

From this definition, we can see what the values for 1! and 0! are.

$$(n-1)! = \frac{n!}{n} \quad \Rightarrow \quad 1! = (2-1)! = \frac{2!}{2} = 1 \qquad and \quad 0! = (1-1)! = \frac{1!}{1} = 1$$

We also notice that when we have ratios of similar factorials we can reduce them as follows.

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$$

Appendix C – Gamma Functions

The gamma function, Γ(x) is defined by the following integral.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad \text{[C-1]}$$

In this definition, t is a dummy variable of integration and the argument of the gamma function, x, appears only in the term t^{x-1}. The gamma function can be shown to be a generalization of factorials. To do this we use equation [C-1] to evaluate Γ(x+1). To do this, we simply replace x by x + 1 everywhere in that equation and integrate by parts to obtain

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x+1-1} dt = \int_0^\infty t^x d(-e^{-t}) = -e^{-t} t^x \Big|_0^\infty - \int_0^\infty \left(-e^{-t}\right) d\left(t^x\right) \qquad \text{[C-2]}$$

$$\int u dv \qquad = uv\Big| \qquad - \int v du$$

If we restrict the value of x to be greater than zero, the term $e^{-t}t^x$ vanishes at both the upper and lower limit. (At the upper limit, we have to apply l'Hopital's rule to get this result.) The $d(t^x)$ term in the integral is simply $xt^{x-1}dt$ and when we perform this differentiation we see that the resulting integral is simply xΓ(x).

$$\Gamma(x+1) = -e^{-t} t^x \Big|_0^\infty - \int_0^\infty \left(-e^{-t}\right) d\left(t^x\right) = 0 - x\int_0^\infty - e^{-t} t^{x-1} dt = x\Gamma(x) \qquad \text{[C-3]}$$

The gamma function for an argument of 1 is particularly simple to compute.

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \int_0^\infty e^{-t} dt = -e^{-t} \Big|_0^\infty = -(0-1) = 1 \qquad \text{[C-4]}$$

We can apply equation [C-3] to this result to obtain the value of the gamma function for 2, 3, and 4 as follows.

$$\Gamma(2)=\Gamma(1+1)=1\Gamma(1)=1 \qquad \Gamma(3)=\Gamma(2+1)=2\Gamma(2)=2 \qquad \Gamma(4)=\Gamma(3+1)=3\Gamma(3)=6$$
[C-5]

We see that the general result of equation [C-5] is

$$\Gamma(n+1)=n\Gamma(n)=n! \qquad \text{[C-6]}$$

We can apply equation [C-3] and the result that Γ(1) = 1 to compute Γ(0).

$$\Gamma(x+1)=x\Gamma(x) \qquad \Rightarrow \qquad \Gamma(x)=\frac{\Gamma(x+1)}{x} \qquad \textit{so that } \Gamma(0)\to\infty \qquad \text{[C-7]}$$

Extending this relationship to negative integers tells us that the value of the gamma function for negative integers is infinite. However, the value of the gamma function for and noninteger number, including negative ones, may be found from equation [C-3], once the values of the gamma function are known in the interval between zero and one. Numerical methods are generally required to find these noninteger values of the gamma function. One exception to this is the value of the gamma function of one-half that can be found by contour integration in the complex plane. The result of such integration is

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \qquad \text{[C-8]}$$

Note that we can use equation [C-3] to get other values of the gamma function such as the following.

$$\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2} \qquad \Gamma\left(\frac{5}{2}\right)=\frac{3\sqrt{\pi}}{4} \qquad \Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi} \qquad \text{[C-9]}$$